

SCATTERING THEORY FOR P-FORMS ON HYPERBOLIC REAL SPACE

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ABSTRACT. Due to spectral obstructions, a scattering theory in the Lax-Phillips sense for the wave equation for differential p -forms on H^{n+1} cannot be developed. As a consequence, Huygens' principle for the wave equation in this context does not hold.

If we restrict the class of forms and we consider the case of coclosed p -forms on H^{n+1} , when $n = 2p$, Huygens' principle does hold and thus in this case incoming and outgoing subspaces can be constructed.

1. INTRODUCTION

Scattering theory for the wave equation has been developed by Lax and Phillips essentially for functions, at first in euclidean setting (see [5] and the references therein), and more recently, after the papers of Faddeev and Pavlov (see [3]) and the construction of a Fourier analysis for functions on symmetric spaces due to Helgason (see [4]), in non-euclidean setting. Particular attention has been devoted in [6], [7], [8], [9] to the wave equation for functions on the hyperbolic space H^{n+1} and on quotients of H^{n+1} by groups of isometries.

The purpose of this paper is to extend the Lax-Phillips theory to the case of the wave equation for p -forms, or more generally for sections of vector bundles, on complete, non compact Riemannian manifolds. The basic motivation is connected, on one hand, with the spectral properties of the Laplace-Beltrami operator on differential forms, and hence in some way with the "geometry" of the manifold, and on the other side with the validity of Huygens' principle for the wave equation, and therefore with the "analysis" that can be developed on the manifold.

A general approach to these problems on a general manifold turns out to be extremely difficult to deal with, for the lack of information about the behaviour of Laplace-Beltrami operator on a generic complete non compact Riemannian manifold, and of analytic tools mimicking the Radon and Fourier transforms.

These facts motivate our choice to investigate the problem at first on the hyperbolic space H^{n+1} . In this case, the spectrum of the Laplace-Beltrami operator on p -forms on H^{n+1} has been explicitly computed by Donnelly

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(see [2]). Moreover in recent years (see [1]) it has been developed a Radon transform for p -forms on H^{n+1} , providing an essential tool to the study of the wave equation.

In section 2, we recall some essential facts about H^{n+1} and its spectral properties. In section 3, we prove that it is not possible to develop a scattering theory in the Lax-Phillips sense for the wave equation for all p -forms on H^{n+1} . In section 4, we show that, as a consequence, Huygens' principle for the wave equation in this context does not hold. In section 5 we show that since Huygens' principle does hold for the wave equation for coclosed p -forms when $n = 2p$, a scattering theory in the Lax-Phillips sense can be developed if we impose some restrictions on the class of forms considered. This result implies some conclusions about the spectrum of the Laplace-Beltrami operator on coclosed p -forms on H^{n+1} , when $n = 2p$.

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2. PRELIMINARIES

We will denote by H^{n+1} the real hyperbolic space of dimension $n + 1$, either in the half-space model

$$H^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_{n+1} > 0\}$$

endowed with the Riemannian metric

$$ds^2 = \frac{dx_1^2 + dx_2^2 + \dots + dx_{n+1}^2}{x_{n+1}^2},$$

or in the Poincaré model

$$H^{n+1} = \{(x_1, \dots, x_{n+1}) \in \mathbb{R}^{n+1} \mid |x|^2 < 1\}$$

with the Riemannian metric

$$d\sigma^2(x) = \left(\frac{2}{1 - |x|^2} \right)^2 (dx_1^2 + \dots + dx_{n+1}^2),$$

where $|x|^2 = x_1^2 + x_2^2 + \dots + x_{n+1}^2$.

We will denote by $\Lambda^p(H^{n+1})$ the space of smooth p -forms on H^{n+1} , and by $\Lambda_c^p(H^{n+1})$ the space of smooth compactly supported p -forms on H^{n+1} . As usual, $L_p^2(H^{n+1})$ will be the Hilbert space of square integrable p -forms on H^{n+1} , that is the completion of $\Lambda_c^p(H^{n+1})$ with respect to the L^2 norm induced by the Riemannian metric:

$$\|\omega\|_{L^2}^2 = \int_{H^{n+1}} \omega \wedge * \omega.$$

Since H^{n+1} is a complete Riemannian manifold, the Laplace-Beltrami operator $\Delta_p : \Lambda_c^p(H^{n+1}) \rightarrow \Lambda_c^p(H^{n+1})$ is an essentially selfadjoint operator in $L_p^2(H^{n+1})$. We will denote its closure again by the same symbol Δ_p .

H. Donnelly in [2] has explicitly computed the spectrum of the Laplace-Beltrami operator on p -forms on H^{n+1} . He has proved that Δ_p has point spectrum if and only if $n+1 = 2p$. In this case, the only eigenvalue of Δ_p is zero, occurring with infinite multiplicity, and the continuous part of the operator is unitarily equivalent to the operator

$$L^2(\mathbb{R}^+, dr, M \oplus M) \longrightarrow L^2(\mathbb{R}^+, dr, M \oplus M)$$

$$f_1(x) \oplus f_2(x) \longmapsto \left(\frac{1}{4} + x^2\right) f_1(x) \oplus \left(\frac{1}{4} + x^2\right) f_2(x),$$

where M is a Hilbert space of countably infinite dimension and dr is the Lebesgue measure; hence, if $n+1 = 2p$, the absolutely continuous spectrum of Δ_p is $\sigma_{ac}(\Delta_p) = [\frac{1}{4}, +\infty)$, with constant multiplicity.

If $p \neq \frac{n+1}{2}$, the point spectrum is empty, and Δ_p is unitarily equivalent to the operator

$$L^2(\mathbb{R}^+, dr, M \oplus M) \longrightarrow L^2(\mathbb{R}^+, dr, M \oplus M)$$

$$f_1(x) \oplus f_2(x) \longmapsto \left(\left(\frac{n-2p}{2}\right)^2 + x^2\right) f_1(x) \oplus \left(\left(\frac{n-2p+2}{2}\right)^2 + x^2\right) f_2(x).$$

Hence $\sigma(\Delta_p) = \sigma_{ac}(\Delta_p) = [\min\{(\frac{n-2p}{2})^2, (\frac{n-2p+2}{2})^2\}, +\infty)$, and the multiplicity of the spectrum is not constant.

In the following we will suppose $p \neq \frac{n+1}{2}$.

3. GENERAL CASE

Consider the Cauchy problem for the wave equation for p -forms on H^{n+1} :

$$(3.1) \quad \begin{cases} \Delta_p \omega + \partial_{tt}^2 \omega = 0 \\ \omega(0) = \omega_1 \\ \partial_t \omega(0) = \omega_2, \end{cases}$$

with ω_1, ω_2 smooth and compactly supported. For every $t \geq 0$

$$(3.2) \quad \langle \Delta_p \omega(t), \omega(t) \rangle + \langle \partial_t \omega(t), \partial_t \omega(t) \rangle = \langle \Delta_p \omega_1, \omega_1 \rangle + \langle \omega_2, \omega_2 \rangle,$$

because

$$\begin{aligned} \partial_t (\langle \Delta_p \omega(t), \omega(t) \rangle + \langle \partial_t \omega(t), \partial_t \omega(t) \rangle) &= \\ &= \langle \Delta_p (\partial_t \omega), \omega \rangle + \langle \Delta_p \omega, \partial_t \omega \rangle + \\ &+ \langle \partial_{tt}^2 \omega, \partial_t \omega \rangle + \langle \partial_t \omega, \partial_{tt}^2 \omega \rangle = 0. \end{aligned}$$

Hence, the energy preserved by the system is given by

$$\|(\omega_1, \omega_2)\|_H^2 = \langle \Delta_p \omega_1, \omega_1 \rangle + \langle \omega_2, \omega_2 \rangle \geq 0,$$

which defines indeed a norm on $\Lambda_c^p(H^{n+1})$, because $\|(\omega_1, \omega_2)\|_H = 0$ implies:

$$\begin{cases} d\omega_1 = 0 \\ \delta\omega_1 = 0 \\ \omega_2 = 0, \end{cases}$$

that is

$$\begin{cases} \Delta_p \omega_1 = 0 \\ \omega_2 = 0. \end{cases}$$

Since $p \neq \frac{n+1}{2}$ and therefore 0 is not in the point spectrum, $\Delta_p \omega_1 = 0$ implies $\omega_1 = 0$, showing in conclusion that

$$\|(\omega_1, \omega_2)\|_H = 0 \implies (\omega_1, \omega_2) = (0, 0).$$

It is therefore meaningful to take the completion H of the space of smooth compactly supported initial data with respect to this latter norm, which is induced by the scalar product

$$(3.3) \quad \langle (\omega_1, \omega_2), (\tilde{\omega}_1, \tilde{\omega}_2) \rangle = \langle \Delta_p \omega_1, \tilde{\omega}_1 \rangle + \langle \omega_2, \tilde{\omega}_2 \rangle.$$

The operator A defined on

$$D(A) = \Lambda_c^p(H^{n+1}) \oplus \Lambda_c^p(H^{n+1})$$

by

$$(3.4) \quad A = \begin{pmatrix} 0 & I \\ -\Delta_p & 0 \end{pmatrix}$$

is essentially skewselfadjoint in H . Hence its closure \overline{A} is the generator of a one-parameter group of unitary operators $U(t)$ on H which maps every initial data into the value at time t of the corresponding solution of the wave equation for p -forms on H^{n+1} .

The operator $i\overline{A}$ is essentially selfadjoint and can be shown to be unitarily equivalent to the operator

$$\begin{pmatrix} \sqrt{\Delta_p} & 0 \\ 0 & -\sqrt{\Delta_p} \end{pmatrix}$$

on $L_p^2(H^{n+1}) \oplus L_p^2(H^{n+1})$ (see [10]).

Now, thanks to Donnelly's result, $\sqrt{\Delta_p}$ is unitarily equivalent to the direct sum of multiplication operators

$$L^2(\mathbb{R}^+, dr, M \oplus M) \longrightarrow L^2(\mathbb{R}^+, dr, M \oplus M)$$

$$f_1(x) \oplus f_2(x) \mapsto \left(\sqrt{\left(\frac{n-2p}{2}\right)^2 + x^2} f_1(x) \oplus \sqrt{\left(\frac{n-2p+2}{2}\right)^2 + x^2} f_2(x) \right).$$

As a consequence, if $n+1 \neq 2p$ the spectrum of $i\overline{A}$ is purely absolutely continuous, equal to

$$\begin{aligned} & \left(-\infty, \max \left\{ -\left| \frac{n-2p}{2} \right|, -\left| \frac{n-2p+2}{2} \right| \right\} \right] \\ & \cup \\ & \left[\min \left\{ \left| \frac{n-2p}{2} \right|, \left| \frac{n-2p+2}{2} \right| \right\}, +\infty \right) \end{aligned}$$

and does not have constant multiplicity.

Now, if it would be possible to develop a scattering theory in the Lax-Phillips sense for the wave equation for p -forms on H^{n+1} , there would exist an incoming or an outgoing translation representation of $(H, U(t) = e^{\overline{A}t})$, that is there would exist a unitary operator

$$R : H \longrightarrow L^2(\mathbb{R}, N),$$

such that

$$RU(t)R^{-1} = T_0(t),$$

where N is an auxiliary Hilbert space and $T_0(t)$ denotes the translation of t units to the right. In turn, this would imply that $i\overline{A}$ has a purely absolutely continuous spectrum on the whole real axis, and moreover that the spectrum of $i\overline{A}$ has constant multiplicity. As a matter of fact, if there would exist an incoming or an outgoing translation representation, $U(t)$ on H would be unitarily equivalent to $T_0(t)$ on $L^2(\mathbb{R}, N)$, hence its generator \overline{A} would be unitarily equivalent to the operator $\frac{d}{dx}$ on $L^2(\mathbb{R}, N)$. As a consequence, $i\overline{A}$ would be unitarily equivalent to $i\frac{d}{dx}$, and hence, through a Fourier transform, to the multiplication operator

$$L^2(\mathbb{R}, N) \longrightarrow L^2(\mathbb{R}, N)$$

$$f(x) \longmapsto xf(x),$$

whose spectrum is purely absolutely continuous, covers the whole real line and has constant multiplicity.

Since we have shown that the multiplicity of the spectrum of $i\overline{A}$ is not constant, we can conclude that there can be no incoming or outgoing translation representation for $(H, U(t))$. Hence:

Theorem 3.1. *It is not possible to develop a scattering theory following the Lax-Phillips approach for all p -forms on H^{n+1} .*

Only in the cases $n = 2p$ or $n = 2p - 2$ one can hope that, restricting in some way the class of forms, it might be possible to develop a scattering theory for the wave equation in the Lax-Phillips sense.

4. HUYGENS' PRINCIPLE

Consider the Cauchy problem (3.1) for the wave equation for p -forms, with smooth initial data ω_1, ω_2 , and fix a point $x_0 \in H^{n+1}$. We will say that Huygens' principle holds if the following conditions are fulfilled:

1. if the initial data are zero in the ball $B_{H^{n+1}}(x_0, R)$ then the solution $\omega(x, t)$ at time t of the initial value problem is zero in the ball $B_{H^{n+1}}(x_0, R - t)$;
2. if the initial data vanish outside the ball $B_{H^{n+1}}(x_0, R)$, then the solution $\omega(x, t)$ at time t of the initial value problem vanishes in the ball $B_{H^{n+1}}(x_0, |t| - R)$.

The spectral properties of the Laplace-Beltrami operator can be used in order to prove that Huygens' principle fails in the case of the wave equation for p -forms on H^{n+1} , when $p \neq \frac{n+1}{2}$.

Consider the Cauchy problem (3.1) for the wave equation for p -forms on H^{n+1} , with initial data ω_1, ω_2 smooth and compactly supported.

As noticed before, the operator A defined on $\Lambda_c^p(H^{n+1}) \oplus \Lambda_c^p(H^{n+1})$ by (3.4) is essentially skewselfadjoint in H endowed with the energy norm, and its closure \overline{A} is the generator of a one-parameter group of unitary operators $U(t)$ on H which describes the solution of the wave equation for p -forms on H^{n+1} .

Now, let D_+ be the subspace of H of those smooth initial data (ω_1, ω_2) such that for every $t > 0$

$$\begin{cases} (d\omega_1)(x, t) = 0 \\ (\delta\omega_1)(x, t) = 0 \\ \omega_2(x, t) = 0 \end{cases}$$

for $d_{H^{n+1}}(x, x_0) < t$, and

$$(\omega_1(x, t), \omega_2(x, t)) = (U(t)(\omega_1, \omega_2))(x).$$

The closure $\overline{D_+}$ of D_+ is a closed subspace of H such that

$$U(t)\overline{D_+} \subseteq \overline{D_+}$$

for every $t > 0$. To show that

$$(4.1) \quad \bigcap_t U(t)\overline{D_+} = \{\underline{0}\}$$

note that, if $(\omega, \eta) \in U(t)\overline{D_+}$, for $t > 0$, there exists a sequence $\{(\omega_n, \eta_n)\} \subset D_+$ such that

$$U(t)(\omega_n, \eta_n) \longrightarrow (\omega, \eta)$$

in H , that is:

$$\begin{cases} d\omega_n \longrightarrow d\omega \\ \delta\omega_n \longrightarrow \delta\omega \\ \eta_n \longrightarrow \eta \end{cases}$$

Now,

$$\begin{cases} (d\omega_n)(x, t) = 0 \\ (\delta\omega_n)(x, t) = 0 \\ \eta_n(x, t) = 0 \end{cases}$$

when $d_{H^{n+1}}(x, x_0) < t$, and this implies:

$$\begin{cases} (d\omega)(x, t) = 0 \\ (\delta\omega)(x, t) = 0 \\ \eta(x, t) = 0 \end{cases}$$

when $d_{H^{n+1}}(x, x_0) < t$. Therefore, if $(\omega, \eta) \in \bigcap_t U(t)\overline{D_+}$, then

$$\begin{cases} (d\omega)(x, t) = 0 \\ (\delta\omega)(x, t) = 0 \\ \eta(x, t) = 0 \end{cases}$$

when $d_{H^{n+1}}(x, x_0) < t$ for every $t > 0$, and this implies $(\omega, \eta) = \underline{0}$ in H , i.e. that 4.1 holds.

Now, if Huygens' principle would hold, then

$$\bigcup_t U(t)\overline{D_+} = H.$$

In fact, if $(\omega, \eta) = 0$ for $d_{H^{n+1}}(x_0, x) > R$, then, by Huygens' principle,

$$U(t)(\omega, \eta) = 0 \text{ for } d_{H^{n+1}}(x_0, x) < t - R$$

for every $t > 0$. In particular,

$$U(t + R)(\omega, \eta) = 0 \text{ for } d_{H^{n+1}}(x_0, x) < t + R - R = t$$

for every $t > 0$. This in turn would imply

$$U(R)(\omega, \eta) \in D_+.$$

Thus, $U(-R)D_+$ would contain the set of all initial data with support in $B_{H^{n+1}}(x_0, R)$, and hence

$$\bigcup_t U(t)D_+ \subseteq \bigcup_t U(t)\overline{D_+}$$

would contain all compactly supported data, which are dense in H .

Therefore, if Huygens' principle would hold, D_+ would be an outgoing subspace for $U(t)$, and, thanks to Sinai's theorem, there would exist an outgoing translation representation for H , $U(t)$. Now, this contradicts the properties of the spectrum of Δ_p . In conclusion:

Theorem 4.1. *Huygens' principle does not hold for the wave equation for p -forms on H^{n+1} , when $p \neq \frac{n+1}{2}$.*

5. COCLOSED CASE

In the case $p = \frac{n}{2}$, the spectrum of the Laplace-Beltrami operator is $[0, +\infty)$, with non constant multiplicity. This fact makes it conceivable that, under some restrictions on the forms considered, a scattering theory in the Lax-Phillips sense could be developed for the wave equation.

This is indeed the case, if the class of coclosed p -forms, with $n = 2p$, in H^{n+1} is considered.

Indeed, it has been shown in [1] that Huygens' principle holds for the wave equation for coclosed p -forms on H^{n+1} , when $n = 2p$.

As a consequence:

Theorem 5.1. *Incoming and outgoing subspaces exist for the wave equation for coclosed p -forms on H^{n+1} , when $p = \frac{n}{2}$.*

Proof. Consider the space of coclosed smooth, compactly supported p -forms in H^{n+1} , with $n = 2p$, and the Cauchy problem (3.1) for the wave equation with coclosed smooth initial data ω_1, ω_2 . The condition

$$\delta\omega_1(x) = \delta\omega_2(x) = 0$$

is equivalent to

$$\forall t \quad \delta\omega(t, x) = \delta\omega_t(t, x) = 0,$$

where $\omega(x, t)$ is the solution of the Cauchy problem corresponding to the initial data ω_1, ω_2 .

Indeed, if the initial data ω_1, ω_2 are coclosed and $\omega(x, t)$ denotes the solution of the corresponding Cauchy problem, then $\delta\omega(x, t)$ is the solution of the Cauchy problem for the wave equation for $(p-1)$ -forms, with initial data $\delta\omega_1, \delta\omega_2$. Now, thanks to the uniqueness of the solution of the Cauchy problem for the wave equation for $(p-1)$ -forms, if $\delta\omega_1 = 0, \delta\omega_2 = 0$ then

$$\delta\omega(x, t) = 0$$

for every t .

Consider now the Cauchy problem for the wave equation in the space of coclosed forms. The energy left invariant by the system is expressed by the norm

$$\|(\omega, \eta)\|_{\tilde{H}}^2 = \langle \Delta_p \omega, \omega \rangle + \|\eta\|^2 = \|d\omega\|^2 + \|\eta\|^2 \geq 0,$$

which is induced by the scalar product (3.3) restricted to the set of coclosed forms. The Hilbert space \tilde{H} , completion of the space D of all smooth, compactly supported coclosed data with respect to this norm, is a closed subspace of H .

We will denote by \tilde{A} the restriction of the operator A , expressed by (3.4), to D . \tilde{A} maps D into D , and is essentially skewselfadjoint in \tilde{H} . Consequently, it is the generator of a one-parameter group of unitary operators $\tilde{U}(t)$, which describes the evolution of the system, mapping initial data to the corresponding solution at time t .

Let now \tilde{D}_+ be the subspace of \tilde{H} of those initial data (ω_1, ω_2) such that for every $t > 0$

$$\begin{cases} (d\omega_1)(x, t) = 0 \\ \omega_2(x, t) = 0 \end{cases}$$

for $d_{H^{n+1}}(x, x_0) < t$, where x_0 is a fixed point of H^{n+1} and

$$(\omega_1(x, t), \omega_2(x, t)) = (\tilde{U}(t)(\omega_1, \omega_2))(x).$$

Its closure $\overline{\tilde{D}_+}$ in \tilde{H} is a closed subspace of \tilde{H} such that

$$\tilde{U}(t)\overline{\tilde{D}_+} \subseteq \overline{\tilde{D}_+}.$$

Moreover, it satisfies

$$\bigcap_t \tilde{U}(t)\overline{\tilde{D}_+} = \{\mathbf{0}\}.$$

Finally, the validity of Huygens' principle implies that

$$\bigcup_t \tilde{U}(t) \overline{\tilde{D}_+} = \tilde{H},$$

because, if Huygens' principle holds, then, for every $R > 0$, $\tilde{U}(-R)\tilde{D}_+$ contains every smooth compactly supported coclosed data with support in $B(x_0, R)$. Hence, $\overline{\tilde{D}_+}$ is an outgoing subspace for $\tilde{U}(t)$ in \tilde{H} . Analogously, the subspace \tilde{D}_- of \tilde{H} , consisting of those initial data (ω_1, ω_2) such that for every $t < 0$

$$\begin{cases} (d\omega_1)(x, t) = 0 \\ \omega_2(x, t) = 0 \end{cases}$$

for $d_{H^{n+1}}(x, x_0) < -t$, can be shown to be an incoming subspace for $\tilde{U}(t)$ in \tilde{H} . \square

Theorem 5.1 implies that the spectrum of $i\overline{\tilde{A}}$ is purely absolutely continuous, with constant multiplicity, and equal to $(-\infty, +\infty)$. As a consequence:

Theorem 5.2. *When $n = 2p$, the spectrum of the Laplace-Beltrami operator on coclosed p -forms on H^{n+1} is purely absolutely continuous, with constant multiplicity, equal to $[0, +\infty)$.*

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